Rotational Motion About a Fixed Axis

Vocabulary

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10-1 Angular Quantities

- In this chapter we will deal with rotational motion
- We will focus on the rigid body rotating about a fixed axis
- A rigid body is a body with a definite shape that doesn’t change, so that the particles that compose it stay in fixed positions relative to one another.
- Any real body is capable of vibrating or deforming when a force is exerted on it
- Effects are small, so the concept of an ideal rigid body is very useful as a good approximation
- The motion of a rigid body can be analyzed as the translational motion of its center of mass plus rotational motion about its center of mass
- Purely rotational motion - all points in the body move in circles and that the centers of these circles all lie on a line called the axis of rotation
- \( R \) will be used to represent the perpendicular distance of a point or particle from the axis of rotation
- We do this to distinguish \( R \) from \( r \), which will continue to represent the position of a particle with reference to the origin (a point) of some coordinate system
- Every point in a body rotating about a fixed axis moves in a circle whose center is on the axis and whose radius is \( R \), the distance of that point from the axis of rotation
- A straight line drawn from the axis to any point sweeps out the same angle \( \phi \) in the same time
- We will use \( \phi \) to distinguish the angle of some particular line in the body with respect to some reference line
- Angles are commonly stated in degrees, but in mathematics of circular motion it is much simpler if we use the radian for angular measure
- One Radian (rad) is defined as the angle subtended by an arc whose length is equal to the radius
- If \( l = R \) then \( \phi \) is exactly equal to 1 rad
Equation Box 10-1

- where R is the radius of the circle
- l is the arc length subtended by the angle \( \phi \) which is specified in radians
- note that the radian is dimensionless since it is the ratio of two lengths
- Radians can be related to degrees in the following way
  - \( l = 2\pi r \)
  - \( \phi = \frac{1}{R} \)
  - \( \phi = \frac{2\pi R}{R} \)
  - So

Equation Box 10-2

- angular velocity (denoted by _________) is defined in analogy with ordinary linear velocity
- Instead of linear displacement, we use angular displacement
- the magnitude of the average angular velocity is defined as

Equation Box 10-3

- where \( \Delta \phi \) is the angle through which the body has rotated in the time \( \Delta t \)
- the magnitude of the instantaneous angular velocity is the limit of this ratio as \( \Delta t \) approaches zero

Equation Box 10-4

- Angular velocity is generally specified in radians per second (rad/s)
- Note that all points in a rigid body rotate with the same angular velocity, since every position in the body moves through the same angle in the same time interval
Angular acceleration \( (\text{____}) \) in analogy to ordinary linear acceleration, is defined as the change in angular velocity divided by the time required to make this change.

- Average angular acceleration
- where \( \omega_1 \) is the angular velocity initially and \( \omega_2 \) is the angular velocity after a time \( \Delta t \)
- Instantaneous angular acceleration is defined as the limit of this ratio as \( \Delta t \) approaches zero
- Since the angular velocity is the same for all points of a rotating body
- Angular acceleration will also be same for all points
- Thus angular acceleration and velocity are properties of the rotating body as a whole
- All rotating rigid bodies also have linear velocity.

\[
\text{Equation Box 10-5}
\]

The linear velocity is greater for points farther from the axis- angular velocity is the same for all point in a rotating rigid body

- You can also have tangential acceleration which is

\[
\text{Equation Box 10-6}
\]

\( R \) is the radius of the circle in which the particle is moving and \( \text{Atan} \) stands for tangential since the acceleration is along the circle
- Earlier we saw that a particle moving in a circle of radius \( R \) with linear speed \( \text{v} \) has a radial or centripetal acceleration

\[
\text{Equation Box 10-7}
\]

- this acceleration points toward the center of the circular path
• This equation applies to any particle of a rotating object
• Thus the centripetal acceleration is greater the farther you are from the axis of rotation (on a merry go-round the furtherest from the center feels the greatest acceleration)

• We can relate the angular velocity to the frequency of rotation, \( f \), where by frequency we mean the number of complete revolutions (rev) per second
• one revolution (of a wheel) corresponds to an angle \( 2\pi \) radians
• \( 1 \text{ rev/s} = 2\pi \text{ rad/s} \)
• \( f \) is related to angular velocity by

• The unit for frequency is (rev/s) and is given a special name- Hertz (Hz)
• \( 1 \text{ Hz} = 1 \text{ rev/s} = 1 \text{ s}^{-1} \)
• Note that “revolution” is not really a unit
• The time required for one complete revolution is the period, \( T \), and it is related to frequency by

10-2 Kinematic Equations for Uniformly Accelerated Rotational Motion
• The definitions of angular velocity and acceleration are the same as their linear counterparts, except that \( \phi \) has replaced the linear displacement \( x \)
• ω has replaced v
• ω has replaced a
• angular equations for constant angular acceleration will be analogous to our equations earlier

<table>
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<tr>
<th>Angular</th>
<th>Linear</th>
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</tbody>
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10-3 Rolling Motion (without slipping)
• Rolling without slipping is readily analyzed and depends on static friction between the rolling object and the ground
• the friction is static because the rolling object’s point of contact with the ground is at rest at each moment
• kinetic friction comes into it if you brake too hard- but this complicates it
• Rolling without slipping involves both rotation and translation
• there is a simple relation between linear speed v of the axle and the angular velocity

Equation Box 10-12

• The linear speed V of the axle relative to the ground is related to the angular velocity by the above equation
• Note that this is valid only if there is no slipping

10-4 Vector Nature of Angular Quantities
• both angular acceleration and velocity are vectors
• the linear velocities of different particles of the wheel point in all different directions
• The only unique direction in space associated with the rotation is along the axis of rotation, perpendicular to the actual motion
• Therefore choose the axis of rotation to be the
direction of the angular velocity vector
• For convection in determine the direction of the vector we will use the right hand rule- then the fingers of the right hand are curled around the rotation axis and point in the direction of the rotation, then the thumb points in the direction of \( \omega \)

• If the axis of rotation is fixed in direction, then \( \omega \) can change only in magnitude
• Just like with linear acceleration and velocity- when the velocity increases then acceleration points in the same direction as velocity- when velocity decreases then acceleration points in the opposite direction as velocity

10-5 Torque
• This is devoted to the dynamics, or causes of rotational motion
• to make an object to start moving there must be a force on it
• Direction of this force and where it is applied are also important
• The angular acceleration is proportional to the magnitude of the force, but also directly proportional to the perpendicular distance from the axis of rotation to the line along which the force acts
• This perpendicular distance is called the lever arm or moment arm
• The force is labeled \( R_1 \) and \( R_2 \) for the two forces
• the angular acceleration is proportional to the product of the force time the lever arm
• this product is called the moment of the force about the axis or more commonly called the torque _____
• Thus the angular acceleration _____ of an object is directly proportional to the net applied torque _____
• we see that it is torque that gives rise to angular acceleration
• the rotational analog of Newton’s second law for linear motion (a related to \( F \))
• So the equation for torque is similar to Newtons Second law

**Equation Box 10-13**

- \( R \) is the lever arm (must use the distance from the axis of rotation that is perpendicular to the line of action of the force
- another way of solving for torque is to resolve it into its parallel and perpendicular components
- so if we look at the perpendicular force then the equation is
if we take into account the angle that is formed by the force, then the equation becomes

Since torque is a distance times a force- it is measured in units of m•N in SI units
When more than one torque acts on a body- the acceleration ___ is found to be proportional to the net torque
If all the torques acting on a body tend to rotate it in the same direction, the net torque is the sum of the torques
but if one torque acts to rotate a body in one direction and the second torque acts to rotate it in an opposing direction, the net torque is the difference of the two torques
you still use the signs like before (+ in one direction- counterclockwise) and vice versa

10-6 Rotational Dynamics; Torque and Rotational Inertia
• remember it is the net torque just like it was the net force and it is proportional to the angular acceleration
• in the linear case the acceleration is not only proportional to the net force but it is also inversely proportional to the inertia of the body, which we call its mass
• so what is the mass for rotational bodies? since it seems that there is a parallel between newton’s second law and torque
• we will take the idea of torque, newton’s second law, and atan and blend together

Then we multiply by R on both sides, the torque is given by
Equation Box 10-17

- single particle
- here we have the direct relation between angular acceleration and the applied torque
- The quantity \( mR^2 \) represents the rotational inertia of the particle and is called its moment of inertia
- now if we sum all the internal torques we get the moment of inertia of the body - I

Equation Box 10-18

- if we combine the equations together 10-17 and 10-18 then we can simplify and write

Equation Box 10-19

- This is the rotational equivalent of Newton’s second law
- It is valid for the rotation of a rigid body about a fixed axis
- it is also valid for a body translating with acceleration as long as I and alpha are calculated about the center of mass of the body and the rotation axis through the cm does not change direction

Equation Box 10-20

- The moment of inertia I which is a measure of the rotational inertia of a body, plays the same role for rotational motion that mass does for translational motion
- Careful- mass can not be considered concentrated at CM for rotational motion

10-7 Solving Problems in Rotational Dynamics
Whenever dealing with torque and angular acceleration, it is important to use a consistent set of units which in SI is \( \text{rad/s}^2 \), torque in \( \text{m} \cdot \text{N} \), and the moment of inertia in \( \text{kg} \cdot \text{m}^2 \). 

- \( I \) depends on axis of rotation and on distribution of mass.
- For most ordinary bodies, the mass is distributed continuously and the calculation of the moment of inertia can be difficult.

<table>
<thead>
<tr>
<th>Object</th>
<th>Location of Axis</th>
<th>Drawing</th>
<th>Moment of Inertia</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thin hoop of radius ( R_0 )</td>
<td>through center</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Thin hoop of radius ( R_0 ) and width ( w )</td>
<td>through center diameter</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Solid cylinder of radius ( R_0 )</td>
<td>through center</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hollow cylinder of inner radius ( R_1 ) and outer radius ( R_2 )</td>
<td>through center</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Uniform sphere of radius ( r_0 )</td>
<td>through center</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Long uniform rod of length ( l )</td>
<td>through center</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Long uniform rod of length ( l )</td>
<td>through end</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rectangular thin plate of length ( l ) and width ( w )</td>
<td>through center</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

When calculation is difficult, \( I \) can be determined experimentally by measuring the angular acceleration about a fixed axis due to a known net torque and applying the equation \( I = \frac{\text{net torque}}{\text{angular acceleration}} \).

**10-8 Determining moments of Inertia**

- **By experiment**
  - The moments of inertia of any body about any axis can be determined experimentally by measuring the net torque required to give the body angular acceleration.
  - **Using Calculus**
• many bodies can be considered as a continuous distribution of mass.
• In this case the moment of inertia becomes an integral

**The Parallel Axis and Perpendicular Theorems**
• Parallel axis theorem - states that \( I \) is the moment of inertia of a body of total mass \( M \) about any axis, and \( I_{cm} \) is the moment of inertia about an axis passing through the center of mass parallel to the first axis but a distance \( h \) away

\[
\text{Equation Box 10-21}
\]

• The parallel axis theorem can be applied to any body
• Perpendicular axis theorem can be applied only to plane figures - to two dimensional bodies or bodies of uniform thickness whose thickness can be neglected compared to the other dimensions
• This theorem states that the sum of the moments of inertia about an axis in the plane of the body is equal to the moment of inertia about an axis through their point of intersection perpendicular to the plane of the object

\[
\text{Equation Box 10-22}
\]

10-9 Angular Momentum and Its Conservation
• just like before we will see that the linear counterpart and the angular counterpart are similar
• Linear momentum is \( p = mv \)
• Angular momentum - \( L \)
• for a body rotating along a fixed axis it is
• \( I \) is the moment of inertia
• \( \omega \) is the angular velocity
• SI units for \( L \) are \( \text{kg} \cdot \text{m}^2/\text{s} \)
• The rotational equivalent of Newton’s second law was and can be written as

\[
\text{Equation Box 10-23}
\]
• The derivative of that equation is valid even if the moment of inertia changes

**Conservation of Angular Momentum**
• this is an important concept because under certain conditions it is a conserved quantity
• If the L does not change then the torque does not change and therefore the angular momentum is conserved

**Law of Conservation of angular momentum- the total angular momentum of a rotating body remains constant if the net external torque acting on it is zero**
• When there is zero net torque acting on a body, and the body is rotating about a fixed axis or about an axis through its cm such that its direction doesn’t change

**Equation Box 10-23**

• Skaters doing a spin on ice, illustrates conservation of angular momentum
  • I is small and angular velocity is small and then vice versa
• for angular momentum to be conserved, the net torque must be zero, but the net force does not necessarily have to be zero

**Vector Nature of Angular Momentum**
• For a body rotating about a fixed axis, the direction of the angular momentum can be taken as the direction of the angular velocity
• This is strictly true only if the rotation axis is an axis of symmetry body or if the body is thin and flat and rotates about an axis perpendicular to the plane of the body
• The right hand rule applies to determining the vectors

10-10 Rotational Kinetic Energy
• the quantity \( \frac{1}{2} mv^2 \) is the kinetic energy of a body undergoing translational motion
• A body rotating about an axis is said to have rotational kinetic energy
• The translational kinetic energy we would expect that the rotational energy be

**Equation Box 10-24**

• this is indeed true
• you can use the idea of kinetic energy and the idea of rotational energy to derive the equation
• Then you can also use the work energy principle for the rotation
• We will not go to this level since it is calculus based

10-11 Rotational Plus translational Motion Rolling

• Kinetic Energy = \( K_{cm} + K_{rot} \)
• an object that rotates while its center of mass (cm) undergoes translational motion will have both translational and rotational kinetic energy
• The total kinetic energy is

\[
\text{Equation Box 10-25}
\]

• If there was no friction then a sphere and other rolling objects would simply slide.
• For calculations we will not take into account the static friction for the energy because it is static friction and it does no work
• The point of contact for a rolling object moves perpendicular to the plane

*Right Hand Rule image*
hyperphysics.phy-astr.gsu.edu